

Index Theorems for Polynomial Pencils

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Abstract

We survey index theorems counting eigenvalues of linearized Hamiltonian systems and characteristic values of polynomial operator pencils. We present a simple common graphical interpretation and generalization of the index theory using the concept of graphical Krein signature. Furthermore, we prove that derivatives of an eigenvector $u = u(\lambda)$ of an operator pencil $\mathcal{L}(\lambda)$ satisfying $\mathcal{L}(\lambda)u(\lambda) = \mu(\lambda)u(\lambda)$ evaluated at a characteristic value of $\mathcal{L}(\lambda)$ do not only generate an arbitrary chain of root vectors of $\mathcal{L}(\lambda)$ but the chain that carries an extra information.

1 Introduction

Spectral problems naturally arise in investigations of stability and decay rates of nonlinear waves and stability of numerical schemes, in solving integrable systems via inverse scattering method, and in multiple other fields. The main interest is in presence of point spectrum with a positive real part (*unstable eigenvalues*) that corresponds to destabilizing modes. Over the last 40 years counts of unstable eigenvalues and other related counts that we refer to as *index theorems* appeared across various distinct and unrelated fields due to their simple structure and importance for applications. Here we briefly survey index theory literature, point out its common graphical interpretation, and derive its generalization to problems with arbitrary structure of the kernel of the operators involved.

Linearized Hamiltonian systems. Index theorems proved to be particularly useful in spectral stability theory of waves in Hamiltonian systems where one studies spectrum $\sigma(JL)$ of the non-selfadjoint problem

$$JLu = \nu u, \quad J = -J^*, \quad L = L^*. \quad (1)$$

Here J and L are operators acting on a Hilbert space X , L is the second variation of the underlying Hamiltonian, $u \in X$, and L^* denotes the adjoint operator of L

[71, 62]. Problem (1) appears in search for exponentially growing or decaying solutions $v(x, t) = e^{\nu t}u(x)$ of the linearized Hamiltonian system $v_t = JLv$ around the equilibrium $v_0 = v_0(x, t)$ of the (nonlinear) Hamiltonian system with $v(x, t)$ representing an infinitesimal perturbation of v_0 . The wave v_0 is said to be spectrally stable if (1) has no solution with $\text{Re } \nu > 0$ for $v \in X$. Due to the natural symmetry of spectrum $\sigma(JL)$ (see [42]) the spectral stability is equivalent to the confinement of $\sigma(JL)$ to the imaginary axis. While positivity of the spectrum of L implies spectral stability, the operator L often has negative eigenvalues, and due to the symmetries of the system also a non-trivial kernel opening up an opportunity for instability in the system. However, the symmetries through the Noether theorem imply existence of conserved quantities (typically corresponding to physically meaningful quantities as mass, momentum, etc.) and their conservation restricts possible degrees of freedom in the system and thus can prohibit instability in cases of indefinite L . The index theorems for linearized Hamiltonians relate the spectra of L and JL , and particularly the number of negative real eigenvalues of L and unstable eigenvalues of JL . Analogous index theory was developed for quadratic operator pencils. The link between these two types of results is that for invertible J it is possible to reformulate (1) as a linear operator pencil. Therefore, both types of index theorems can be viewed as special cases of a general theory for operator pencils.

Two different ways to interpret the index theorems mathematically can be found in the literature. Motivated by the work of Hestenes [28] (see also [20]) Maddocks [58, 60] derived the dimension counts for restricted quadratic forms in finite dimensional spaces and showed how the question of stability of an equilibrium of a Hamiltonian system reduces to a question whether a quadratic form is positive when restricted to a particular subspace of its domain. Such an approach was later used in works [24, 66, 32, 15, 27, 4] and it is closely related to the theory of indefinite inner product spaces [7, 29, 54]. It provides a geometric visualization of the index theorems as counts of dimensions of an intersection of a negative energy cone associated with the indefinite quadratic form with the subspace spanned by normal vectors (under the indefinite inner product) to hyperplanes tangential to surfaces of conserved quantities (Fig. 1, left panel).

However, here we focus on an alternative point of view of a different geometrical (graphical) nature [5, 41, 42]. We interpret the index theorems as topological counts of curves of eigenvalues of operator pencils in a plane (Fig. 1, right panel). We believe that such an interpretation provides not only a simpler visualization of the theory but also an easier way to generalize it. More surprisingly, as we will show, it also yields reduced algebraic formulae for calculation of the indices of operators with complicated generalized kernels because the chain of root vectors generated by the graphical method carry an extra information compared to a (generic) chain of root vectors.

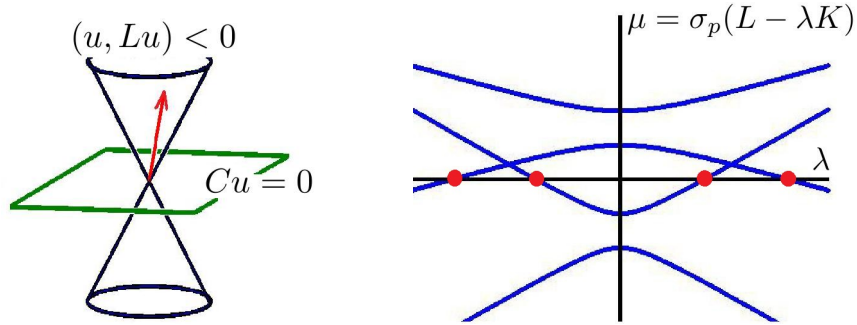


Figure 1: Visualization of the index theory. Left panel: Algebraic approach. The equilibrium is (spectrally) stable if and only if the normal vector (in the associated indefinite inner product space) to the plane $Cu = 0$ corresponding to the invariant (conserved quantity) of the system lies in the negative energy cone $\mathcal{C} = \{u \in X, (u, Lu) < 0\}$ [58]. Right panel: Graphical approach. Plot of eigenvalue branches $\mu = \mu(\lambda)$ (the point spectrum $\sigma_p(L - \lambda K)$) of the eigenvalue pencil $(L - \lambda K)u = \mu u$. Purely imaginary eigenvalues ν of JL correspond to intercepts of $\mu(\lambda)$ with the axis $\mu = 0$ via $\lambda = i\nu$ (indicated by full circles). Their Krein signature is given by the sign of $\mu'(\lambda)$ [42].

2 Krein Signature

Index theorems often refer to *Krein signature* of eigenvalues (characteristic values), a quantity that characterizes ability of the (stable) eigenvalue to become unstable under a perturbation [48]. In linearized Hamiltonian systems the Krein signature $\kappa_L(\lambda)$ of an eigenvalue ν of JL captures the signature of the quadratic form $(\cdot, L \cdot)$ representing the linearized energy on the invariant subspace spanned by eigenspaces corresponding to $(\nu, -\bar{\nu})$ (see MacKay [57] for the geometric visualization). Krein signature is also referred to as *sign characteristics* [19] within the context of operator pencils and *symplectic signature* in Hamiltonian mechanics (see Kirillov [38] for a detailed discussion of the terminology, literature survey, and extension of the results of [57]). If the signature of the energy on the subspace is indefinite, the Krein signature is said to be *indefinite*, otherwise it is *definite* (*positive* or *negative*). It is easy to see that the Krein signature of non-purely imaginary and non-semi-simple purely imaginary eigenvalues is indefinite [68, 29]. On the other hand, the signature of any simple non-zero purely imaginary eigenvalue of (1) is definite [43].

If J is invertible it is useful to define $K = (iJ)^{-1}$ and to note that K is self-adjoint. Then $Lu = i\nu Ku$ for a simple purely imaginary eigenvalue ν of JL with the eigenvector u and

$$\kappa_L(\nu) := \text{sign}(Lu, u) = \text{sign } i\nu(u, Ku).$$

Thus the sign of (u, Ku) agrees with $\kappa_L(\nu)$ up to the sign of $i\nu$ and one can

define

$$\kappa_K(\lambda) := -(u, Ku), \quad \text{for } \lambda := i\nu, \quad Lu = \lambda Ku. \quad (2)$$

Since the definition (2) is closely related to its graphical analogue, we will drop the index K in (2) in the rest of the paper with exceptions at places where it is necessary to avoid a confusion. Due to the rotation $\lambda = i\nu$ the main interest lies in Krein signature of $\lambda \in \mathbb{R}$, so we limit ourselves to a definition of Krein signature of real eigenvalues of (1). An analogous definition can be given for the Krein indices and signature of a real characteristic value of a polynomial self-adjoint operator pencil [19, 42].

Definition 1. Let J be an invertible skew-adjoint and L be a self-adjoint operator on the Hilbert space X and let λ_0 be a real eigenvalue of iJL with \mathcal{U} one of its corresponding chains of generalized eigenvectors. Let $K = (iJ)^{-1}$ and let W be the (Hermitian) Gramm matrix of the quadratic form $(\cdot, -K \cdot)$ on the span of \mathcal{U} . The number of positive (negative) eigenvalues of W is called the positive (negative) Krein index of \mathcal{U} at λ_0 and is denoted $\kappa^+(\mathcal{U}, \lambda_0)$ ($\kappa^-(\mathcal{U}, \lambda_0)$). The sums of $\kappa^\pm(\mathcal{U}, \lambda_0)$ over all (maximal) chains of generalized eigenvectors \mathcal{U} of λ_0 are called the positive and negative Krein indices of λ_0 and are denoted $\kappa^\pm(\lambda_0)$. Finally, $\kappa(\mathcal{U}, \lambda_0) := \kappa^+(\mathcal{U}, \lambda_0) - \kappa^-(\mathcal{U}, \lambda_0)$ is called the Krein signature of the chain \mathcal{U} for λ_0 , and $\kappa(\lambda_0) := \kappa^+(\lambda_0) - \kappa^-(\lambda_0)$ is called the Krein signature of λ_0 .

Consider the spectrum of the operator pencil $\mathcal{L}(\lambda) := L - \lambda K$, i.e., the set of $\mu = \mu(\lambda)$ for which there exists $u \in X$ such that

$$\mathcal{L}(\lambda)u = \mu u. \quad (3)$$

The real point spectrum of iJL corresponds one-to-one (including multiplicity) to the set of real characteristic values of $\mathcal{L}(\lambda)$, i.e., to $\lambda_0 \in \mathbb{R}$ such that $\mathcal{L}(\lambda_0)$ has a non-trivial kernel [19, 61, 42]. Under suitable assumptions the eigenvalues $\mu(\lambda)$ and eigenvectors $u(\lambda)$ can be chosen to be real analytic in λ [42] and it is possible to define the graphical Krein signature for a self-adjoint operator pencil [42, 61].

Definition 2. Let $\mathcal{L}(\lambda)$ be a self-adjoint operator pencil and assume that \mathcal{L} has an isolated real characteristic value λ_0 and there are real analytic eigenvalue branches $\mu(\lambda)$ of $\mathcal{L}(\lambda)$ such that eigenvalues of $\mathcal{L}(\lambda)$ for λ close to λ_0 are identical to $\mu(\lambda)$. Let $\mu = \mu(\lambda)$ be one of the branches with $\mu^{(n)}(\lambda_0) = 0$ for $n = 0, 1, \dots, m-1$, and $\mu^{(m)}(\lambda_0) \neq 0$. Let $\eta(\mu) := \text{sign}(\mu^{(m)}(\lambda_0)) = \pm 1$. Then the quantities

$$\kappa_g^\pm(\mu, \lambda_0) := \begin{cases} \frac{1}{2}m, & \text{for } m \text{ even,} \\ \frac{1}{2}(m \pm \eta(\mu)), & \text{for } m \text{ odd,} \end{cases} \quad (4)$$

are called the positive and negative graphical Krein indices of the eigenvalue branch $\mu = \mu(\lambda)$ corresponding to the characteristic value λ_0 . The sums of $\kappa_g^\pm(\mu, \lambda_0)$ over all eigenvalue branches crossing at $(\lambda, \mu) = (\lambda_0, 0)$ are called

the positive and negative graphical Krein indices of λ_0 and are denoted $\kappa_g^\pm(\lambda_0)$. Finally, $\kappa_g(\mu, \lambda_0) := \kappa_g^+(\mu, \lambda_0) - \kappa_g^-(\mu, \lambda_0)$ is called the graphical Krein signature of the eigenvalue branch $\mu = \mu(\lambda)$ vanishing at λ_0 , and $\kappa_g(\lambda_0) := \kappa_g^+(\lambda_0) - \kappa_g^-(\lambda_0)$ is called the graphical Krein signature of λ_0 .

Definition 2 extends to general self-adjoint operator pencils as long as smooth eigenvalue and eigenvector branches exist in a neighborhood of an isolated characteristic value λ_0 [42]. The fundamental relation between the Krein signature and the graphical Krein signature for real λ [42, 61, 5] is given by

$$\kappa_g(\lambda) = \kappa_K(\lambda). \quad (5)$$

The Krein signature $\kappa(\lambda)$ then can be read off the graph of spectrum of $\mathcal{L}(\lambda)$ and the chain of generalized eigenvectors of iJL corresponding to λ_0 can be generated by derivatives of the eigenfunction branch $u(\lambda)$ corresponding to the eigenvalue $\mu(\lambda)$ of (3) [61, 42, 41].

3 Index Theorems for Linear Pencils and Linearized Hamiltonians

We introduce the following notation. Let X and Y be separable Hilbert spaces and let A be a densely defined operator $D(A) \subset X \rightarrow Y$. We denote $\sigma_p(A)$ the point spectrum of A and $n_{\text{uns}}(A)$ the unstable index of A counting the number of unstable eigenvalues of A , i.e. the number of points in $\sigma_p(A) \cap \{\text{Re}(z) > 0\}$. Furthermore, let $p(A)$, $z(A)$, and $n(A)$ be, respectively, the counts of, respectively, positive, zero, and negative real eigenvalues of a self-adjoint operator A counting (geometric) multiplicity, allowing the counts to be infinite.

In 1972 Vakhitov and Kolokolov [74] studied spectral stability of stationary (in an appropriate reference frame) solutions ϕ_ω of a nonlinear Schrödinger equation parameterized by angular velocity ω . Their linearized stability is characterized by the spectrum of the eigenvalue problem (1). The *Vakhitov-Kolokolov criterion* states that if L_\pm are self-adjoint operators, L_+ non-negative, $\sigma(L_-) \cap \mathbb{R}^- = \{-\lambda_0\}$,

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad (6)$$

then

$$n_{\text{uns}}(JL) = n(L) - n(dI/d\omega). \quad (7)$$

Here $n(L) = 1$ and $I(\omega) = \int \phi_\omega^2 dx$ is the energy of the stationary solution, i.e., $dI/d\omega$ is a scalar. The quantity $dI/d\omega$ can be related to the sign of the derivative $D'(\lambda)$ at $\lambda = 0$ of the Evans function [65] and to the quadratic form $(\cdot, L \cdot)$ evaluated at the generalized kernel of JL corresponding to the eigenvector ϕ_ω . Thus the count of unstable eigenvalues of JL depends on the number related to $\text{gKer}(JL)$.

Under the assumption of the full Hamiltonian symmetry $\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\}$ of $\sigma_p(JL)$ it is straightforward to generalize (7) to the parity index theorem: $n_{\text{uns}}(JL) - (n(L) - n(D))$ is an even and non-positive integer, where $n(D)$ is the count of negative eigenvalues of a matrix D related to $\text{gKer}(JL)$ (see Theorem 1). This parity index theorem was proved in 1987 in celebrated papers [24, 25] by Grillakis, Shatah, and Strauss who also proved the connection of the spectral stability to nonlinear stability for a wide class of problems (see also [70, Chapter 4] and [67] for the survey of the stability results in the context of nonlinear Schrödinger and KdV equations). The generalization of the parity index theorem for the operators related to stability of dispersive waves was proved by Lin [55]. The parity index theorem plays an important role in the theory of gyroscopic stabilization where it states that if the degree of instability (negative index) of the system is odd the equilibrium point cannot be stabilized by gyroscopic force. This fundamental theorem is often referred to as Thomson theorem [72, 12, 47], Thomson-Tait-Chetaev theorem [63, Chapter 6] and Kelvin's theorem [45] (Lord Kelvin's original name was William Thomson). See the works of Kozlov [44, 46], Kozlov and Karapetyan [47], Chern [13] for further results, references, and discussion of its implications, and Kozlov [45] for the topological implications of the theorem.

It is easy to identify $n_{\text{uns}}(JL) = k_r + 2k_c$, where k_r is the number of positive real eigenvalues of JL and k_c is the number of eigenvalues of JL in $\{z \in \mathbb{C}, \text{Re}(z) > 0, \text{Im}(z) > 0\}$. Furthermore, denote

$$k_i^- = \sum_{\nu \in \sigma_p(JL) \cap i\mathbb{R}, i\nu < 0} \kappa_L^-(\nu) = \sum_{\lambda \in \sigma_p(iJL) \cap \mathbb{R}^-} \kappa_K^-(\lambda).$$

In the case of the operator JL with all purely imaginary eigenvalues simple the number k_i^- counts the purely imaginary eigenvalues of JL with positive imaginary part of negative Krein signature. The final form of the index theorem for linearized Hamiltonians was proved in 2004 independently by Kapitula, Kevrekidis and Sandstede [32] and Pelinovsky [66] (some assumptions of [66] were removed in [75]).

Theorem 1 ([32, 66]). *Let J be an invertible skew-adjoint and L a self-adjoint operator acting on a Hilbert space X , with J^{-1} bounded on a subspace of X of a finite codimension, $n(L) < \infty$, and $\sigma(L) \cap \{x \in \mathbb{R}, x \leq 0\} \subset \sigma_p(L)$. Assume that the operators J and L satisfy (symmetry) assumptions that imply the full Hamiltonian symmetry of $\sigma_p(JL)$ and the fact that all Jordan chains corresponding to kernel of JL have length two. Let $\mathcal{V} = \text{gKer}(JL) \ominus \text{Ker}(L)$ and let D be the (symmetric) matrix of the quadratic form $(\cdot, L \cdot)$ restricted to \mathcal{V} . Then*

$$n_{\text{uns}}(JL) = k_r + 2k_c = n(L) - n(D) - 2k_i^-. \quad (8)$$

Note that the count (8) under the assumption $z(L) = 0$ was already established in 1988 by Binding and Browne [5, Proposition 5.5] in the context of

Sturm-Liouville theory. They considered the case of L semi-bounded with compact resolvent and J 1-to-1 and used the standard perturbation theory combined with the graphical Krein signature referred to as *two parameter spectral theory*. Their beautiful short and simple argument is based on graphical inspection of eigenvalues branches (eigencurves) that can be interpreted as a homotopy in the parameter μ_0 from $L + \mu_0 \mathbb{I}$ positive definite to L indefinite and counting of the eigenvalue branches intersections with the axis $\mu = 0$ (see also [6]). Furthermore, the claim and the proof of the Theorem 1 is in some extent implicitly present in the theory developed by Iohvidov [29], Langer [54], and also in Bogner [7, Section XI.4].

For the operators arising in the spectral theory of Sturm-Liouville problems with indefinite weight an index theorem (8) plays an important role. While the upper bound of $n_{\text{uns}}(JL)$ is well understood [79, Theorem 5.8.2], the exact count and dependence of its individual factors on the coefficient of the underlying differential equation poses an important open problem [79, Problems IX–X, p. 300, Problem 1, p. 124, Comment (7), p. 128].

Kollár and Miller [42] proved the index theorem for selfadjoint matrix pencils that we further generalize in Section 4 and that implies a generalization of a finite-dimensional version of Theorem 1. For periodic Hamiltonian systems Kapitula and Hărăguș [27] proved the analogue of (8) using the Floquet theory (Bloch wave decomposition) under technical assumptions related to the Keldysh theorem that guarantees completeness of the eigenvectors for (1). Some of the technical assumptions of [27] were later removed in [17]. See also [10] for an alternative proof of (8) based on the integrable structure of the underlying problem. Recently, Stefanov and Kapitula [34] and Pelinovsky [67] removed the assumption of boundedness of J^{-1} and proved (8) for the case covering the KdV-type problems with $J = \partial_x$ under the assumption $\dim(\text{Ker } L) = 1$ (see [67] for historical discussion of the stability results). Chugunova and Pelinovsky [15] studied the generalized eigenvalue problem $Lu = \lambda Ku$ using the theory of indefinite inner product spaces and particularly Pontryagin Invariant Subspace Theorem and proved counts (inertia laws) analogous to (8). Furthermore they showed how (1) can be treated within that context and provided an alternative proof of (8).

In 1988 Jones [30] and Grillakis [22] independently proved the index theorem bounding the number of unstable real eigenvalues of (1) from below for the systems with the canonical form (6).

Theorem 2 ([30, 22]). *Let J and L have the canonical structure (6) with L_{\pm} self-adjoint on a Hilbert space X , $\text{Ker}(L_+) \perp \text{Ker}(L_-)$, and let V denote the orthogonal complement of $\text{Ker}(L_+) \oplus \text{Ker}(L_-)$ in X with the orthogonal projection $P : Y \rightarrow V$. Then*

$$n_{\text{uns}}(JL) \geq k_r \geq |n(PL_+P) - n(PL_-P)|. \quad (9)$$

The proofs in [22] and in [30] are significantly different, with the method of Grillakis [22] related to the graphical Krein signature. Note that Theorem 2

does not rely on completeness of the eigenvectors of JL . Theorem 2 is frequently used to establish instability of various nonlinear waves, particularly in situations when negative spectra and kernel of L_{\pm} are explicitly known.

Kapitula and Promislow [33] reproved Theorem 1 using the theory of [58] for constrained Hamiltonian systems and the Krein matrix theory, and reformulated (1), (6) by inverting the operator L^+ reducing (1) to a generalized eigenvalue problem for which they established (9). They also proved a local count theorem analogous to Theorem 4 of Section 4. Note that Theorem 2 can be also easily obtained as a corollary of a general result of [42] (see Section 4) by using the same reformulation as in [33]. Both counts (8) and (9) were in a more general context also derived by Cuccagna *et al.* in [16] in the setup allowing eigenvalues embedded in the essential spectrum under some further technical assumptions. A lower bound for the number of real eigenvalues for Hermitian matrix pencils was derived by Lancaster and Tismenetsky [52] together with various other index theorems for perturbed Hermitian matrix pencils (including the upper bound for n_{uns}). Also, see Grillakis [23] for the analysis of the case $n(PL_+P) = n(PL_-P)$.

Quadratic eigenvalue pencils and properties of their spectra are a well-studied subject with a large number of applications (see [18, 73], and references therein). Particular areas where index theorems naturally appear are Sturm-Liouville problems [2] and gyroscopic stabilization. The general framework for a study of gyroscopic stabilization and quadratic operator pencils in general is to study spectrum of the pencil $\mathcal{L}(\lambda) = \lambda^2 A + \lambda(D + iG) + K + iN$ where the coefficients A, D, G, K, N are self-adjoint operators (see [37, 63, 46] and references therein) under various additional conditions for the coefficients. A survey of all important results in this area exceeds the scope of this paper and thus we list here only a few of references. Fundamental results for quadratic operator pencils were obtained by Krein and Langer [49, 50] and later extended by Adamyan and Pivovarchik [1] who also proved an index theorem similar to (8). Results that can be expressed in a form of an index theorem were obtained also by Lancaster and his coworkers [51, 53], by Wimmer [77], and Chern [13] (see also reference therein). Important index theorems for systems with dissipation $D > 0$ and partial dissipation $D \geq 0$ were proved in [78, 76, 44].

In the field of stability of nonlinear waves the index theorems for quadratic eigenvalue pencils are a fairly new subject. Chugunova and Pelinovsky [14] proved the count analogous to (8) for the quadratic Hermitian matrix pencils of the form $\lambda^2 \mathbb{I} + \lambda L + M$, where M has either zero or one dimensional kernel (under a further non-degeneracy condition) via an application of the Pontryagin Invariant Subspace Theorem. Their results were later reproved and extended by Kollár [41] and Kollár and Miller [42] (see Example 1 of this manuscript). Recently, Bronski, Johnson and Kapitula [11] proved the count of a similar form to (8) for the quadratic operator pencils $\mathcal{L}(\lambda) = A + \lambda B + \lambda^2 C$, where A and C are self-adjoint and B is invertible skew-symmetric extending results of [64], [69], and [56].

Specific counts of eigenvalues for a particular class of Sturm-Liouville operators given by JL with $J = \text{sign}(x)$ and $L = -d^2/dx^2 + V(x)$ in $L^2(\mathbb{R})$ were obtained in [35, 2]. Various types of index and eigenvalue localization theorems

for definite and indefinite Sturm-Liouville problems, and particularly those that correspond to defective symmetric operators, can be found in [4, 3] and in multiple reference therein, see also Binding and Volkmer [6] where non-real spectra of JL is studied with the use of graphical Krein signature. Further bounds particularly related to graphical Krein signature and eigenvalue branches $\mu(\lambda)$ (see Section 4) were derived in [5]. The local count referred to as the *Krein oscillation theorem* was proved within the context of index theorems by Kapitula [31] using the Krein matrix theory. An infinitesimal version of the (local index) Theorem 4 for matrices is proved in [19, Theorem 12.6].

The theorem guaranteeing existence of a sequence of eigenvalues converging to zero for a general class of operator pencils with compact self-adjoint non-negative coefficients was proved in [41] by a simple homotopy argument as a generalization of the results of [26] (see also references therein). A homotopy argument was also used by Maddocks and Overton [59] to prove the index theorem for dissipatively perturbed Hamiltonian system. Index theorems within the context of isoperimetric calculus of variations were proved in [21]. Bronski and Johnson [9] derived an index theorem for the Faddeev-Takhtajan problem by an approach analogous to work of Klaus and Shaw [39, 40] on the Zakharov-Shabat system. Also, Kozlov and Karapetyan [47] established the index theorem for finite dimensional Hamiltonian systems that bounds the stable index of the system from below and connected the result to gyroscopic stabilization.

To enclose the historical review of results on index theorems let us point out that an unusually large part of the work mentioned within this section can be traced back to the University of Maryland at College Park, where many of the papers were written and many of the ideas were born. J. H. Maddocks, I. Gohberg, L. Greenberg, C. K. R. T. Jones, M. Grillakis, R. L. Pego, and one of the authors of this manuscript (R. K.) were among the others who were involved in the development of the theory.

4 Graphical Interpretation of Index Theorems

The purpose of this section is to derive index theory that encompasses Theorems 1 and 2 and demonstrates their graphical nature. While Theorem 3 was derived in [42] our main results contained in Theorems 4 and 5 generalize the theory developed in [42]. The analysis is for simplicity performed for matrix pencils although the results under specific assumptions can be generalized to infinitely dimensional setting [5].

Definition 3. Let $\mathcal{L} = \mathcal{L}(\lambda)$ be a Hermitian matrix pencil real analytic in λ with a real characteristic value λ_0 . Let $Z_{\lambda_0^-}^\downarrow = Z_{\lambda_0^-}^\downarrow(\mathcal{L})$ (respectively $Z_{\lambda_0^+}^\downarrow = Z_{\lambda_0^+}^\downarrow(\mathcal{L})$) denote the number (counting multiplicity) of eigenvalue curves $\mu = \mu(\lambda)$ of \mathcal{L} with $\mu(\lambda_0) = 0$ and $\mu(\lambda) < 0$ for all $\lambda < \lambda_0$ (respectively for $\lambda > \lambda_0$) sufficiently close to $\lambda = \lambda_0$. Similarly, let $Z_{-\infty}^\downarrow = Z_{-\infty}^\downarrow(\mathcal{L})$ and $Z_{+\infty}^\downarrow = Z_{+\infty}^\downarrow(\mathcal{L})$, respectively, denote the number (counting multiplicity) of eigenvalue curves $\mu =$

$\mu(\lambda)$ of \mathcal{L} with $\mu(\lambda) < 0$ for all $\lambda < 0$ (and $\lambda > 0$ respectively) with sufficiently large $|\lambda|$.

Although it is traditional to consider polynomial Hermitian matrix pencils the theory generalizes in a straightforward manner to real analytic Hermitian matrix pencils $\mathcal{L}(\lambda)$, i.e., real analytic functions $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ for which $\mathcal{L}(\lambda)$ is a Hermitian matrix for each $\lambda \in \mathbb{R}$.

Theorem 3. *Let $\mathcal{L}(\lambda)$ be a real analytic Hermitian matrix pencil. Then the following counts hold*

$$\left(Z_{-\infty}^{\downarrow} + Z_{+\infty}^{\downarrow} \right) - 2n(\mathcal{L}(0)) - \left(Z_{0+}^{\downarrow} + Z_{0-}^{\downarrow} \right) = \sum_{\substack{\lambda < 0 \\ \lambda \in \sigma(\mathcal{L})}} \kappa(\lambda) - \sum_{\substack{\lambda > 0 \\ \lambda \in \sigma(\mathcal{L})}} \kappa(\lambda), \quad (10)$$

$$\left(Z_{-\infty}^{\downarrow} - Z_{+\infty}^{\downarrow} \right) + \left(Z_{0+}^{\downarrow} - Z_{0-}^{\downarrow} \right) = \sum_{\substack{\lambda \neq 0 \\ \lambda \in \sigma(\mathcal{L})}} \kappa(\lambda). \quad (11)$$

Proof. Consider the (parameter dependent) eigenvalue problem (3). According to the perturbation theory [36] the eigenvalue and eigenvector curves $\mu(\lambda)$ and $u(\lambda)$ are analytic in λ . Furthermore, let Q_{\pm} be the quadrants in the (λ, μ) -plane (see Fig. 2). A simple count of curves entering and leaving Q_{\pm} yields the counts

$$Z_{-\infty}^{\downarrow}(\mathcal{L}) - n(\mathcal{L}(0)) - Z_{0-}^{\downarrow}(\mathcal{L}) - \sum_{\lambda < 0, \lambda \in \sigma(\mathcal{L})} \kappa(\lambda) = 0, \quad (12)$$

$$Z_{+\infty}^{\downarrow}(\mathcal{L}) - n(\mathcal{L}(0)) - Z_{0+}^{\downarrow}(\mathcal{L}) + \sum_{\lambda > 0, \lambda \in \sigma(\mathcal{L})} \kappa(\lambda) = 0, \quad (13)$$

Then the sum and the difference of (12) and (13) give (10) and (11). \square

A local version of the index theorem can be proved analogously.

Theorem 4. *Let $\mathcal{L}(\lambda)$ be a real analytic Hermitian matrix pencil. Then the following local index theorem holds for any real λ_1, λ_2 with $\lambda_1 < \lambda_2$:*

$$n(\mathcal{L}(\lambda_1)) - n(\mathcal{L}(\lambda_2)) + Z_{\lambda_1}^{\downarrow} - Z_{\lambda_2}^{\downarrow} = \sum_{\substack{\lambda_1 < \lambda < \lambda_2 \\ \lambda \in \sigma(\mathcal{L})}} \kappa(\lambda). \quad (14)$$

It is easy to see that

$$Z_{\lambda_0}^{\downarrow} - Z_{\lambda_0}^{\downarrow} = Z_{\lambda_0}^{\uparrow} - Z_{\lambda_0}^{\uparrow} = \kappa(\lambda_0), \quad (15)$$

since the eigenvalue branches vanishing at λ_0 at even order do not contribute to the right hand side of (15). Then

$$Z_{\lambda_0}^{\downarrow} + Z_{\lambda_0}^{\downarrow} = \kappa(\lambda_0) + 2Z_{\lambda_0}^{\downarrow}, \quad Z_{\lambda_0}^{\downarrow} + Z_{\lambda_0}^{\uparrow} = Z_{\lambda_0}^{\downarrow} + Z_{\lambda_0}^{\uparrow} = z(\mathcal{L}(\lambda_0)). \quad (16)$$

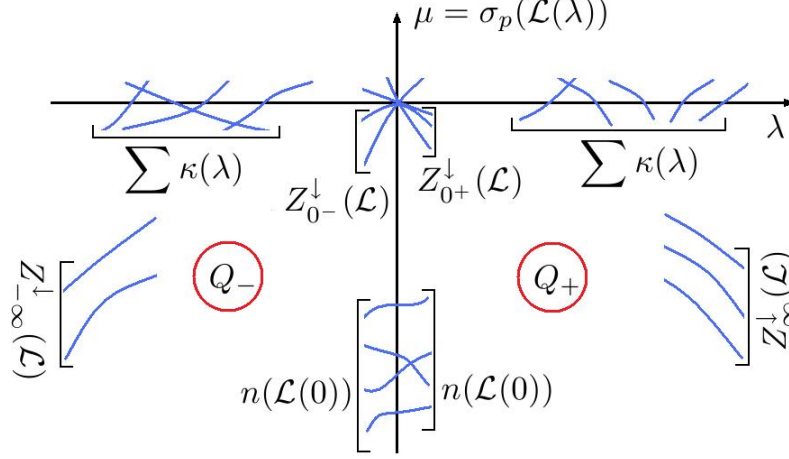


Figure 2: Schematic plot of the proof of Theorem 3. Spectrum $\sigma_p(\mathcal{L}(\lambda))$ organized in eigenvalue branches $\mu = \mu(\lambda)$ vs. λ .

Moreover, if the matrix pencil $\mathcal{L}(\lambda)$ has an extra structure then the terms $Z_{\pm\infty}^\downarrow(\mathcal{L})$ can be determined by the perturbation theory. Let

$$\mathcal{L}(\lambda) = L(\lambda) - g(\lambda)\mathbb{I}, \quad L(\lambda) = \sum_{k=0}^p \lambda^k L_k, \quad g(\lambda) = \sum_{k=0}^q \lambda^k g_k. \quad (17)$$

Here L_0, \dots, L_p are complex Hermitian matrices $n \times n$ and g_0, \dots, g_q are real constants. There is a freedom of choice in an inclusion of identity multipliers in $g(\lambda)$ and $L(\lambda)$ but the index theorems only depend on the leading order term of $\mathcal{L}(\lambda)$ and only differences of $g(\lambda)$ and $L(\lambda)$ are relevant. Therefore we ignore such an ambiguity in (17) and assume $\lambda^p L_p \neq \lambda^q g_q$. Since $\sigma(L(\lambda)) \approx \lambda^p \sigma(L_p)$ and $g(\lambda) \approx \lambda^q g_q$ for $|\lambda| \rightarrow \infty$ the values of terms $Z_{\pm\infty}^\downarrow$ in (10) and (11) depend on the leading order coefficients of $g(\lambda)$ and $L(\lambda)$.

Theorem 5. *Let $\mathcal{L}(\lambda)$ be a real analytic Hermitian matrix pencil satisfying (17) and let L_p is invertible. Then*

p, q	g_q	$Z_{-\infty}^\downarrow(\mathcal{L})$	$Z_{+\infty}^\downarrow(\mathcal{L})$
$q > p, q \text{ even}$	$g_q > 0$	n	n
$q > p, q \text{ even}$	$g_q < 0$	0	0
$q > p, q \text{ odd}$	$g_q > 0$	0	n
$q > p, q \text{ odd}$	$g_q < 0$	n	0
$q < p$		$n((-1)^p L_p)$	$n(L_p)$
$q = p$		$n((-1)^p (L_p - q_p \mathbb{I}))$	$n(L_p - q_p \mathbb{I})$

Setting $g(\lambda) = 0$ in Theorems 3 and 5 one can recover generalizations of Theorems 1 and 2 in a finite dimensional case [42]. Next we illustrate how

specific counts for quadratic eigenvalue pencils can be derived from Theorems 3 and 5.

◁ *Example 1.* Consider the quadratic Hermitian matrix pencil $\mathcal{L}(\lambda)$, $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$

$$\mathcal{L}(\lambda) = M + \lambda K + \lambda^2 \mathbb{I}, \quad M^* = M, \quad K^* = K. \quad (18)$$

Then, $L(\lambda) = M + \lambda K$, $p = 1$, and $g(\lambda) = -\lambda^2$, $q = 2$, and $g_q = -1$ in (17). Therefore $Z_{-\infty}^\downarrow = Z_{+\infty}^\downarrow = 0$ and Theorem 3 implies

$$2n(M) + (Z_{0+}^\downarrow + Z_{0-}^\downarrow) = \sum_{\lambda \in \sigma(\mathcal{L})} \text{sign}(\lambda) \kappa(\lambda), \quad Z_{0+}^\downarrow - Z_{0-}^\downarrow = \sum_{\substack{\lambda \neq 0 \\ \lambda \in \sigma(\mathcal{L})}} \kappa(\lambda).$$

The symmetry $(\lambda, \bar{\lambda})$ of spectrum of \mathcal{L} implies

$$2n = z(\mathcal{L}) + n_r(\mathcal{L}) + 2n_c(\mathcal{L}) + n_i(\mathcal{L}), \quad (19)$$

where $n_r(\mathcal{L})$, $n_i(\mathcal{L})$ and $n_c(\mathcal{L})$ are, respectively, the numbers of real, purely imaginary, and non-real non-purely-imaginary characteristic values of $\mathcal{L}(\lambda)$, $n_i(\mathcal{L})$ is even. Also

$$\begin{aligned} n_r(\mathcal{L}) &= \sum_{\lambda \in \sigma(\mathcal{L}), \lambda > 0} [\kappa^+(\lambda) + \kappa^-(\lambda)] + \sum_{\lambda \in \sigma(\mathcal{L}), \lambda < 0} [\kappa^+(\lambda) + \kappa^-(\lambda)] \\ &= \sum_{\lambda \in \sigma(\mathcal{L})} \text{sign}(\lambda) \kappa(\lambda) + 2 \sum_{\lambda \in \sigma(\mathcal{L}), \lambda > 0} \kappa^-(\lambda) + 2 \sum_{\lambda \in \sigma(\mathcal{L}), \lambda < 0} \kappa^+(\lambda). \end{aligned} \quad (20)$$

Finally, we denote

$$n_r^-(\mathcal{L}) := 2 \sum_{\lambda \in \sigma(\mathcal{L}), \lambda > 0} \kappa^-(\lambda), \quad n_r^+(\mathcal{L}) := 2 \sum_{\lambda \in \sigma(\mathcal{L}), \lambda > 0} \kappa^+(\lambda), \quad (21)$$

motivated by the case of simple real characteristic values of \mathcal{L} where n_r^- , respectively n_r^+ , counts the number of characteristic values of \mathcal{L} with the negative, respectively positive, Krein signature. Then (19) can be rewritten as

$$2n(M) + (Z_{0+}^\downarrow + Z_{0-}^\downarrow) = n_r(\mathcal{L}) - n_r^-(\mathcal{L}) - n_r^+(\mathcal{L}). \quad (22)$$

The symmetric case. If the eigenvalue problem $\mathcal{L}(\lambda)u = \mu u$ has an additional symmetry $(\lambda, \mu) \rightarrow (-\lambda, \mu)$ the index theorem can be further simplified. Clearly, $Z_{0+}^\downarrow = Z_{0-}^\downarrow$, $\kappa^-(\lambda) = \kappa^+(\lambda)$, i.e., $n_r^-(\mathcal{L}) = n_r^+(\mathcal{L})$, and $n_r^+(\mathcal{L}) + n_r^-(\mathcal{L}) = n_r(\mathcal{L})$. Furthermore, all the numbers $z(\mathcal{L})$, $n_r(\mathcal{L})$, $n_c(\mathcal{L})$ are even. A difference of (19) and (22) yields

$$\left[n - \frac{z(\mathcal{L})}{2} \right] - \left[n(M) + Z_{0+}^\downarrow \right] = n_c(\mathcal{L}) + \frac{n_i(\mathcal{L})}{2} + n_r^-(\mathcal{L}) \quad (23)$$

The equation complementary to (23) with respect to (19) is

$$\left[n - \frac{z(\mathcal{L})}{2} \right] + \left[n(M) + Z_{0+}^\downarrow \right] = n_c(\mathcal{L}) + \frac{n_i(\mathcal{L})}{2} + n_r^+(\mathcal{L}) \quad (24)$$

In the special case of a real Hermitian M and purely imaginary Hermitian L , and under the assumption $\text{Ker } M \subset \text{Ker } L$ the counts (23)–(24) correspond to index theorems derived in [14] and [41]. \triangleright

4.1 Algebraic Calculation of Z^\downarrow and Z^\uparrow

The counts (10), (11), and (14) contain terms Z^\downarrow, Z^\uparrow that have a simple graphical interpretation. However, it is more traditional to express them in an algebraic form that we derive in this section. Theorem 6 generalizes the relation between Z^\downarrow and $n(D)$ in Theorem 1 (see [42]) that holds in the case of all Jordan blocks of the eigenvalue 0 of JL of length two to the case of arbitrary structure of the kernel of $\mathcal{L}(\lambda_0)$.

Let $\mathcal{L} = \mathcal{L}(\lambda)$ be a real analytic self-adjoint operator pencil and let λ_0 be its characteristic value of a finite multiplicity. Denote $\mathcal{U}_0 = \text{Ker}(\mathcal{L}(\lambda_0))$, $\dim \mathcal{U}_0 = k$. Then there exist k eigenvalue curves $\mu_j(\lambda)$ and associated eigenvector branches $u_j(\lambda)$, $1 \leq j \leq k$, of $\mathcal{L}(\lambda)$ such that $\mu(\lambda_0) = 0$. Let us define for $m \geq 0$

$$\begin{aligned} K_m^+ &:= \{\mu_i(\lambda); 1 \leq i \leq k, \mu_i^{(s)}(\lambda_0) = 0, 0 \leq s \leq m-1, \mu^{(m)}(\lambda_0) > 0\}, \\ K_m^- &:= \{\mu_i(\lambda); 1 \leq i \leq k, \mu_i^{(s)}(\lambda_0) = 0, 0 \leq s \leq m-1, \mu^{(m)}(\lambda_0) < 0\}, \\ K_m^0 &:= \{\mu_i(\lambda); 1 \leq i \leq k, \mu_i^{(s)}(\lambda_0) = 0, 0 \leq s \leq m\}, \end{aligned} \quad (25)$$

Then the sets K_m^+, K_m^- and K_m^0 are disjoint for any given $m, m \geq 0$, and

$$K_m^0 = K_{m+1}^- \cup K_{m+1}^+ \cup K_{m+1}^0, \quad m \geq 0. \quad (26)$$

For a characteristic value λ_0 of $\mathcal{L}(\lambda)$ of a finite multiplicity $K_m^0 = \emptyset$ for m large enough. Then

$$Z_{\lambda_0}^\downarrow = \left| \bigcup_{m=1}^{\infty} K_m^- \right| = \sum_{m=1}^{\infty} |K_m^-|, \quad Z_{\lambda_0}^\uparrow = \left| \bigcup_{m=1}^{\infty} K_{2m-1}^+ \cup K_{2m}^- \right| = \sum_{m=1}^{\infty} |K_{2m-1}^+| + |K_{2m}^-|.$$

Also, observe that

$$n(\mathcal{L}(\lambda_0)) = |K_0^-|, \quad |\text{gKer}(\lambda_0)| = \sum_{m=1}^{\infty} m (|K_m^+| + |K_m^-|). \quad (27)$$

The quantities $|K_m^\pm|$ can be calculated as the numbers of positive (negative) eigenvalues of specific matrices defined in Theorem 6. Particularly for m odd, $|K_m^\pm|$ count the number of eigenvector chains of λ_0 with positive (negative) Krein index. In that case the Krein index $\kappa(\mathcal{U}, \lambda_0)$ can be calculated by two different ways, either from the (algebraic) definition or by using the graphical Krein signature.

\triangleleft *Example 2.* Let $\mathcal{L}(\lambda) = M + \lambda L + \lambda^2$ be a quadratic Hermitian matrix pencil and let $\mathcal{U} = \{u^{[0]}, u^{[1]}, u^{[2]}\}$ be a chain of root vectors of a characteristic value

$\lambda_0 = 0$ of $\mathcal{L}(\lambda)$. According to the definition of the Krein indices analogous to Definition 1 (see [42] for details) the quantities $\kappa^\pm(\mathcal{U}, \lambda_0)$ of \mathcal{U} count the number of positive and negative eigenvalues of the Gramm matrix W

$$W_{ij} = (u^{[i-1]}, Lu^{[j-1]}) + (u^{[i-2]}, u^{[j-1]}) + (u^{[i-1]}, u^{[j-2]}),$$

for $i, j = 1, 2, 3$, where we formally set $u^{[-1]} = 0$. The characteristic polynomial $f(\lambda) = \det(W - \lambda \mathbb{I})$ is a cubic polynomial with negative leading order coefficient and with three real roots, either two positive and one negative or one positive and two negative. Thus

$$\kappa(\mathcal{U}, 0) = -\text{sign } f(0) = -\text{sign}(\det W). \quad (28)$$

▷

◁ *Example 3.* Consider the quadratic pencil in Example 2 and its characteristic value $\lambda_0 = 0$ and the chain of root vectors $\mathcal{U} = \{u^{[0]}, u^{[1]}, u^{[2]}\}$. According to [42, Theorem 3.4] there exist eigenvalue and eigenvector branches $\mu(\lambda)$, $u(\lambda)$ of (3) such that $\mu(0) = \mu'(0) = \mu''(0) = 0$ and $u(0) = u^{[0]}$. Also $\kappa(\mathcal{U}, 0) = \text{sign } \mu'''(0) \neq 0$. For notational ease we denote $\mathcal{L} = \mathcal{L}(0)$, $u = u(0)$, $\mu = \mu(0)$, and analogously the derivatives ($\mu' = \mu'(0)$, $u' = u'(0)$, etc.). We normalize $(u(0), u(0)) = 1$, differentiate (3) three times, and multiply it by u to obtain

$$\left(u, \frac{\mathcal{L}'''}{3!}u\right) + \left(u, \frac{\mathcal{L}''}{2!}u'\right) + \left(u, \mathcal{L}'\frac{u''}{2!}\right) + \left(u, \mathcal{L}\frac{u'''}{3!}\right) = \frac{\mu'''}{3!}. \quad (29)$$

Since \mathcal{L} is Hermitian the last term on the left hand side of (29) vanishes. Also, differentiation of (3) implies $\mathcal{L}u' + \mathcal{L}'u = 0$ and $\mathcal{L}u'' + 2\mathcal{L}'u' + \mathcal{L}''u = 0$. The operator $\mathcal{L} = \mathcal{L}(0)$ is not invertible but $\mathcal{L} + \Pi$, where Π is the orthogonal projection $X \rightarrow \text{Ker } \mathcal{L}(0)$, is. We denote $\tilde{\mathcal{L}}^{-1} = -(\mathcal{L} + \Pi)^{-1}$ and note that $\tilde{\mathcal{L}}^{-1}\Pi = -\Pi$. Then

$$u' = \tilde{\mathcal{L}}^{-1}\mathcal{L}'u + \Pi u', \quad \text{and} \quad \frac{u''}{2!} = \tilde{\mathcal{L}}^{-1}\mathcal{L}'u' + \tilde{\mathcal{L}}^{-1}\frac{\mathcal{L}''}{2!}u + \Pi\frac{u''}{2!}. \quad (30)$$

Simple algebra (see Theorem 6 for the derivation in the general case) reduces (29) to

$$\kappa(\mathcal{U}, \lambda_0) = \text{sign } \mu'''(0) = \text{sign} [(u, \Lambda_3 u) - (\Pi u', L\Pi u')], \quad (31)$$

where Λ_3 is defined in (34). In the case of the quadratic pencil

$$\Lambda_3 = (M + \Pi)^{-1}L + L(M + \Pi)^{-1} + L(M + \Pi)^{-1}L(M + \Pi)^{-1}L.$$

▷

Note. The formula (31) has important consequences. It contains only $u = u^{[0]}$ and $u' = u'(0)$, i.e., it does not require knowledge of the full Jordan chain \mathcal{U} , contrary to (28). Also the term $(\Pi u', L\Pi u')$ is, generally, non-vanishing and since $\Pi u' \in \text{Ker } \mathcal{L}$, its value is not directly encoded in a (generic) chain \mathcal{U}

as the generalized eigenvectors are determined uniquely only up to a multiple of $u^{[0]}$. It means that $u'(0)$ is not just an arbitrary generalized root vector to $u^{[0]}$ but it captures an extra information $\Pi u'$ that is not, in general, contained in $u^{[1]}$. Thus the chain of root vectors $(u(0), u'(0), u''(0)/2)$ is exceptional that will be also confirmed in a general case of an arbitrary multiplicity. A similar calculation in the case of a quadratic matrix pencil can be found in [8].

As it was illustrated in the Example 3 the graphical approach requires a proper definition of the inverse of the operator $\mathcal{L}(\lambda_0)$. If $\mathcal{L}(\lambda_0)$ is Fredholm and self-adjoint $\text{Ker } \mathcal{L}(\lambda_0) \perp \text{Ran } \mathcal{L}(\lambda_0)$. The operator $\mathcal{L}(\lambda_0)$ acts on the vector Hilbert space $X = \text{Ker } \mathcal{L}(\lambda_0) \oplus \text{Ran } \mathcal{L}(\lambda_0)$. Let $(v_1, v_2) \in \text{Ker } \mathcal{L}(\lambda_0) \oplus \text{Ran } \mathcal{L}(\lambda_0)$ then $\mathcal{L}(\lambda_0)(v_1, v_2) = (0, \mathcal{L}(\lambda_0)v_2)$. The operator $\mathcal{L}(\lambda_0)$ is 1-to-1 on $\text{Ran } \mathcal{L}(\lambda_0)$ and thus the operator $\mathcal{L}(\lambda_0) + \Pi$ is invertible as $(\mathcal{L}(\lambda_0) + \Pi)(v_1, v_2) = (v_1, \mathcal{L}(\lambda_0)v_2)$.

Definition 4. Let $\mathcal{L} = \mathcal{L}(\lambda)$ be an operator pencil acting on a Hilbert space X with a characteristic value λ_0 of a finite multiplicity, and let $\mathcal{L}(\lambda_0)$ has a Fredholm index zero. Let Π be an orthogonal projection $X \rightarrow \text{Ker } \mathcal{L}(\lambda_0)$. Then we define

$$\tilde{\mathcal{L}}^{-1} := -(\mathcal{L}(\lambda_0) + \Pi)^{-1}. \quad (32)$$

Clearly $\mathcal{L}\Pi = 0$ and $\tilde{\mathcal{L}}^{-1}\Pi = -\Pi$. Also, denote $D := d/d\lambda$.

Theorem 6. Let $\mathcal{L} = \mathcal{L}(\lambda)$ be an operator pencil with the characteristic value λ_0 of a finite multiplicity, and $\mathcal{L}(\lambda_0)$ of a Fredholm index zero such that the eigenvalues and eigenvectors of $\mathcal{L}(\lambda)$ in an open real neighborhood of λ_0 is contained in analytic branches $\mu_j(\lambda)$, $u_j(\lambda)$, $j = 1, \dots, k$. Let K_m^\pm , $m \geq 0$, be defined as above and let U_0 be a matrix with vectors in $\text{Ker } \mathcal{L}(\lambda_0)$ as its columns and define recurrently

$$U_{m+1} := U_m \text{Ker}(U_m^* H_{m+1} U_m), \quad (33)$$

where the operator H_m , $m \geq 1$ is defined as $H_m := \Lambda_m + \mathcal{D}_m$. Here

$$\Lambda_m := \sum_{|\alpha|=m} \frac{\mathcal{L}^{(\alpha_1)}(\lambda_0)}{\alpha_1!} \tilde{\mathcal{L}}^{-1} \frac{\mathcal{L}^{(\alpha_2)}(\lambda_0)}{\alpha_2!} \tilde{\mathcal{L}}^{-1} \dots \tilde{\mathcal{L}}^{-1} \frac{\mathcal{L}^{(\alpha_s)}(\lambda_0)}{\alpha_s!}, \quad (34)$$

$$\mathcal{D}_m := \sum_{|\alpha|=m} D^{\alpha_1} \Pi \Lambda_{\alpha_2} \Pi D^{\alpha_3}, \quad (35)$$

where the multi-index $\alpha = (\alpha_1, \dots, \alpha_s)$ has positive integer entries and its norm is calculated as $|\alpha| = \sum_{i=1}^s \alpha_i$. Then for $m \geq 1$

$$|K_m^+| = p(U_{m-1}^* H_m U_{m-1}), \quad |K_m^-| = n(U_{m-1}^* H_m U_{m-1}), \quad U_m^* H_{m+1} U_{m+1} = 0.$$

Proof. We prove Theorem 6 by mathematical induction for $m \geq 1$. Without the loss of generality assume $\lambda_0 = 0$.

First, let $m = 1$. Let $\mu_i \in K_0^0$. Then

$$(\mathcal{L}(\lambda) - \mu_i(\lambda))u_i(\lambda) = 0. \quad (36)$$

Differentiation of (36) with respect to λ at $\lambda = \lambda_0$ where $\mu_i(0) = 0$ and a scalar product with u_j , $u_j \in K_0^0$ yields

$$(u_j, (\mathcal{L}' - \mu'_i)u_i) + (u_j, \mathcal{L}u'_i) = 0. \quad (37)$$

where for a notational ease we drop the argument of \mathcal{L} , μ and u and their derivatives. The second term in (37) vanishes as $(u_j, \mathcal{L}u'_i) = (\mathcal{L}u_j, u'_i) = 0$. Therefore

$$(u_j, \mathcal{L}'u_i) = \mu'_i(u_j, u_i) = \mu'_i \delta_{ij}, \quad (38)$$

with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Consequently, the matrix $(u_j, \mathcal{L}'u_i)$, $u_i, u_j \in \text{Ker } \mathcal{L}(0)$ for $j = 1, \dots, k$, is diagonal with its eigenvalues on a diagonal. The number of its positive, negative and zero eigenvalues is independent of a choice of basis of $\text{Ker } \mathcal{L}(0)$ and it is the signature of the quadratic form $(\cdot, \mathcal{L}'\cdot)$ on $\text{Ker } \mathcal{L}(0)$. Since $\Lambda_1 = \mathcal{L}'(0)$ and $\mathcal{D}_1 = 0$ we derived

$$|K_1^+| = p(U_0^* H_1 U_0), \quad |K_m^-| = n(U_0^* H_1 U_0).$$

Let $h \in \text{Ker}(U_0^* H_1 U_0)$. Then $U_0^* H_1 U_0 h = 0$ and if $v \in U_0 \text{Ker}(U_0^* H_1 U_0) = U_1$ also $U_0^* H_1 U_1 = 0$.

Now assume that the statement of the theorem holds for all j , $j \leq m$. Let $\mu_i \in K_{m-1}^0$. Then differentiation of (36) with respect to λ m -times at $\lambda = 0$ where $\mu_i(0) = \dots = \mu_i^{(m-1)}(0) = 0$ gives

$$\sum_{j=0}^m \frac{\mathcal{L}^{(m-j)}}{(m-j)!} \frac{u_i^{(j)}}{j!} = \frac{\mu_i^{(m)}}{m!} u_i. \quad (39)$$

Also,

$$\sum_{j=0}^s \frac{\mathcal{L}^{(s-j)}}{(s-j)!} \frac{u_i^{(j)}}{j!} = 0 \quad (40)$$

for $s = 1, \dots, m-1$. Adding the term $\Pi u_i^{(s)}/s!$ to both sides of (40) and inverting the operator $(\mathcal{L} + \Pi)$ yields an expression for $u_i^{(s)}$. Now we rewrite (39) as

$$\frac{\mathcal{L}^{(m)}}{(m)!} u_i + \sum_{j=1}^m \frac{\mathcal{L}^{(m-j)}}{(m-j)!} \frac{u_i^{(j)}}{j!} = \frac{\mu_i^{(m)}}{m!} u_i. \quad (41)$$

and recursively express each term $u_i^{(s)}/s!$, $1 \leq s < m$, that does not contain the projection operator Π until all the terms in the sum on the right hand side contain either u_i or a projection operator Π . Since the total number of derivatives of \mathcal{L} and u_i at $\lambda = 0$ in each term is equal m , and all possible decompositions of m in to a sum are used, equation (41) reduces to

$$\Lambda_m u_i - \sum_{s=1}^{m-1} \Lambda_{m-s} \Pi \frac{u_i^{(s)}}{s!} + \mathcal{L} \frac{u_i^{(m)}}{m!} = \frac{\mu_i^{(m)}}{m!} u_i. \quad (42)$$

Multiplying (42) by $u_j \in K_m^0$ yields

$$(u_j, \Lambda_m u_i) - \sum_{s=1}^{m-1} (u_j, \Lambda_{m-s} \Pi \frac{u_i^{(s)}}{s!}) + (u_j, \mathcal{L} \frac{u^{(m)}}{m!}) = \frac{\mu_i^{(m)}}{m!} \delta_{ij}. \quad (43)$$

Since $u_j \in K_m^0$, it can be also expressed for any $p < m$ as (compare to (42))

$$\Lambda_p u_j - \sum_{s=1}^{p-1} \Lambda_{p-s} \Pi \frac{u_j^{(s)}}{s!} + \mathcal{L} \frac{u_j^{(p)}}{p!} = 0. \quad (44)$$

Then each individual term in the summand in the second term on the left-hand side can be reduced

$$\begin{aligned} (u_j, \Lambda_{m-s} \Pi \frac{u_i^{(s)}}{s!}) &= (\Lambda_{m-s} u_j, \Pi \frac{u_i^{(s)}}{s!}) \\ &= \sum_{r=1}^{m-s-1} (\Lambda_{m-s-r} \Pi \frac{u_j^{(r)}}{r!}, \Pi \frac{u_i^{(s)}}{s!}) - (\mathcal{L} \frac{u_j^{(p)}}{p!}, \Pi \frac{u_i^{(s)}}{s!}) = \sum_{r=1}^{m-s-1} (\Lambda_{m-s-r} \Pi \frac{u_j^{(r)}}{r!}, \Pi \frac{u_i^{(s)}}{s!}) \end{aligned}$$

Therefore using the fact $(u_j, \mathcal{L} \frac{u^{(m)}}{m!}) = 0$ equation (43) can be rewritten as

$$(u_j, H_m u_i) = (u_j, \Lambda_m u_i) + (u_j, \mathcal{D}_m u_i) = \frac{\mu_i^{(m)}}{m!} \delta_{ij}. \quad (45)$$

Therefore the matrix $U_m^* H_m U_m$ is diagonal and

$$|K_{m+1}^+| = p(U_m^* H_m U_m), \quad |K_{m+1}^-| = n(U_m^* H_m U_m), \quad |K_{m+1}^0| = z(U_m^* H_m U_m).$$

Multiplication of (45) by $h \in \text{Ker}(U_m^* H_m U_m)$ gives $U_m^* H_m U_m h = 0$ that implies $U_m^* H_m U_{m+1} = 0$. \square

Note. According to (35) we have $\mathcal{D}_1 = \mathcal{D}_2 = 0$. Also, on the left hand side of (24) are two terms $z(\mathcal{L})$ and Z_{0+}^\downarrow that are connected with the properties of (generalized) kernel of \mathcal{L} . It is easy to see that under an assumption $\text{Ker}(M) \subset \text{Ker}(L)$ one has $z(\mathcal{L}) = 2 \dim(\text{Ker}(M))$ that leads to the simplified expression in (8) (see [41] for the details).

5 Conclusions

We presented a unifying view of the index theorems frequently used across various fields. Furthermore, we demonstrated a special property of the chain of root vectors generated by the graphical method that allowed us to derive formulae for the number of eigenvalue curves of the eigenvalue problem $\mathcal{L}(\lambda)u = \mu u$ entering the lower half-plane of the plane (μ, λ) through the characteristic value λ_0 of $\mathcal{L}(\lambda)$ that did not require knowledge of the full chain of the root vectors. Both these results demonstrate the extraordinary beauty and power of the graphical approach. Let us conclude by a quote from [6]: “*Eigencurves* (produced by the graphical approach) *seem to provide a very useful tool in a variety of circumstances, and their theory and applications are quite underdeveloped*”, a statement that certainly remains true even today.

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